UNIVERSITY OF PENNSYLVANIA

RELATIVITY AND THE CRITICAL PHILOSOPHY

FRANK KASSEL

A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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PREFACE

The following pages present the thesis that the Theory of Relativity sets up a definition of physical or ateleological law, by a method which extends the Kantian conception of an equipment presupposed by and applied to experiment. This equipment may therefore be characterized as both *apriori* and *synthetic*.

It is proper that the writer express his indebtedness to the University of Pennsylvania and to the Sage School of Philosophy at Cornell University for supplying the means and freedom to carry on the work. In physics he desires Prof. E. H. Kennard to accept his gratitude for guidance in the formal treatment of the restricted theory and the improvement upon Mach's definition of mass. At the University of Pennsylvania the writer acknowledges his obligations to Prof. Edgar A. Singer, Jr., in the spirit of whose philosophy the composition was written.

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I

The Problem

Kant's space and time are forms which the material of experience assumes, as wine assumes the shape of the wine glass. Instead of "identifying" the individual descriptively by an enumeration of its qualities, they would "locate" the individual by means of their own coordinates. And since for Kant the laws of space are Euclid's laws, for him all sense material must fall into Euclid's mould, if it is to become an object of experience at all.

Since Kant's day, several geometries have been formulated which, denying one or more axioms of Euclid, yet remain self-consistent systems. Of these several systems, equally possible, only one may be actual: can the question, *which one*, be answered, save by an appeal to experience? To those who consider Kant to have held that a geometry to be apriori must be Euclidean, an affirmative answer seems to negate all Kant's teaching on the subject of space.

But may it not be, Kant's geometry was Euclidean only in the sense his mechanics was Newtonean; namely, in the historical sense that these happened to be the only models presenting themselves to his imagination? If so, need the historical accidents that confined the imagination of the man, Kant, limit the scope of that theory of knowledge which goes by the name of *Critical Philosophy?*

In truth, it would seem feasible to maintain that these more general geometries, and particularly their application in the *Theory of Relativity*, confirm rather than confute what is essential to the *Critical Theory* of space, time, and reality.

For Kant, the essential task of this philosophy was to determine how much of our science comes to us from without, how much we ourselves contribute to that fabric of experience we call the objective world. Both this problem-setting and the motives of its solution are best understood as an attempt to expose and correct the errors inherent in certain precedent schools of thought.

Rationalism had ended by regarding the laws of nature as "eternal truths" based on definition, the only type of statement whose denial involved contradiction. From this barren philosophy of "mere meanings," *Empiricism* had appealed to the "fact of experience," as to that which alone could guarantee *existence.* For Empiricism, "things" were given, "laws" were found.

The *Critical* contention was, that *things* became objects for the experimenter only by virtue of *laws* already embodied in them. The essence of the Kantian "revolution" lay in its conclusion, that the investigating mind could find (some) uniformities *in* nature only after it had imposed (other) uniformities *on* nature.

Consider from the point of view of Criticism what prepossessions enter into the "observing" of any familiar object. The diverse qualities and shapes of which we become aware as from different angles we view a table are combined into an individual object—*this table.* But its *unity* is not "given," as are the patches of color, fragments of extension, etc., however indispensable it may be to the experiencing of these patches, fragments, etc., as an individual object. Just as the law of a mathematical series is not itself a term of the series, so the combination of sensations into a synthetic whole is not itself a sensation. Prerequisite as it is to the experiencing of a manifold of sensations as an object in space and time, this unification can be nothing but a contribution of mind to objective experience: it is the apriori factor in science.

The source (Grund) of this unification (Verbindung) of sense-qualities, Kant finds in the "transcendental unity of apperception;" the *one-ness* of the many qualities is the work of the *I* which thinks of them as one. But the *one-ness* of "this table" promptly reveals a two-fold sense: it is *one* with the unity common to all objects, however variously composed of qualities; and it is *one* again as *this* table is *one-other than that* table however like it in quality. Both kinds of one-ness are conferred on objects by the mind that "takes them in"; but as Kant sees the matter, they are conferred by the mind in different capacities:

insofar as it grasps many qualities as *some* object, the mind is said to recognize, *to understand;* in so far as it distinguishes the object as *this* and not *that*, it is said to individuate, *to sense.* It is in attempting to explain how the experiencing mind can individuate, and at what point in the process of experiencing its individuating begins, that Kant comes on his doctrine of apriori space-time.

Kant's theory of individuation is "extensional" (or "denotative"), as opposed to the "intensional" (or "connotative") method of Rationalism. For a Rationalist like Leibnitz, two leaves of the forest to be distinct must ultimately differ in some quality, if by no more than the shadow of a shade. (For an Empiricist like Locke, to be sure, two specimens indistinguishable in quality may yet remain two by virtue of that material substance in which their qualities inhere. But then, for Locke, this substance is "something I know not what.") In the same sense that the Law recognizes an "alibi" to be a complete defense against an "identification by description," however minute, Kant insists that not comparison of qualities, but location in space-time constitutes the mind's "principle of individuation." (So far, obviously, nothing confines Kant to the space-time frame of Euclid and Galileo: he is merely opposing space-time coordinates *of some kind* to description *of any kind.*)

So much for the first stage in Kant's theory. The second and more important one carries him on to the thesis, that just because space-time is the observer's means of individuating, the observer must from the very outset of his experimental task possess this means, and use it. To demonstrate the Critical motives for so daring a doctrine, it will be most convenient to analyze "experience" in its scientific form; i. e., experience as deliberate experiment. Take, for example, a measurement of length: ordinarily the experimenter determines the length of a given rod by applying the rod to his standard-scale, then taking the indicated reading. And this he does not once but twice, thrice, several times. Of these "readings" he takes the arithmetic mean, and by well-known methods calculates a "mean-

variation" (or, better, the more complicated "probable error"). It is easy to see that this repetition of readings proceeds from no mere counsel of carefulness and caution, it is essential to the very meaning of "mean-reading" and "probable error". But these two values are indispensable to the experimenter's definition of that "real" to which his "observation" refers, and yet from which it is distinguished. For his "real" is nothing but the "limiting conception" (as Kant himself might have said) of a mean-reading to which *zero* probable-error attaches. Moreover, just as essential to this meaning of "the mean" as is a multiplicity of readings, must be the assurance *that the readings have been made on the selfsame object.* The same *kind* of object (an object assuming, however minutely, the same description as the original) will not do at all as a substitute for the original.

Conclusion: Whatever is indispensable to assuring the identity of an individual thing, is indispensable to the beginning of an experimental investigation: it must be possessed *prior to* the experiment, and knowledge of it is presupposed by the very process by which empirical knowledge is obtained. But space-time is the only apparatus of individuation. Ergo, space-time must be brought by the experimenter to his experiment, he cannot wait for the experiment to give it to him.

But if this analysis correctly displays the essential motives (as opposed to the historical accidents of expression) of the Critical doctrine, may we not risk a further observation? Namely, as there was nothing in the first part of the argument to confine this philosophy to the Euclidean as to the one space-time "form" capable of individuating; neither is there aught in the second part to prevent this philosophy from considering a question on whose answer turns our whole interpretation: Granted that the mind must bring *one* space-time form to any given experiment, may not this mind be moved by the outcome of that very experiment to bring *another* form to bear on its further experimenting?

As to the effect this reflective interpretation of the historic doctrine would have upon its particular tenet, "that the axioms of geometry are apriori synthetic judgments"; perhaps it would

be this: Kant certainly allowed himself to pick out specifically Euclidean postulates (e. g. " two straights can intersect in only one point") as examples of such apriori synthetic judgments. But could he have waited till Riemann (say) had written, we may imagine him putting his thought in another way; namely, (1) *If* this is an axiom of the geometry a given stage of empirical science has brought to its experiments, *then* for that science it is apriori; (2) And to say that the mind has brought to its experience a form characterized by this axiom, is a synthetic judgment. Having put the matter in this way, it may be doubted whether Kant would feel that he had changed his philosophy; only his historic setting, his fashion of imagination, his range of illustration would have been altered.

II

Neo-Geometries

But the value of these last reflections cannot be judged until we have considered the motives of those who would see in the " neo-geometries" the refutation of all that was most essentially Critical. If, we have admitted, the space-time which Kant held to be the condition of experimenting is unique in kind (and that kind characterized by such postulates of Euclid as Kant himself cites) then is his philosophy undone by the non-Euclidean geometries. There are those who can see no alternative to this interpretation of post-Kantian history: "The words of Lobatschewsky [writes Bonola] (1, 92) afford evidence of a philosophical conception of space opposed to that of Kant, then generally accepted. The Kantian doctrine considered space as a subjective intuition, a necessary presupposition of every experience. Lobatschewsky's doctrine was rather allied to sensualism and the current empiricism, and compelled geometry to take its place again among the experimental sciences." Now " the words of Lobatschewsky" were these: " The fruitlessness of the attempts made since Euclid's time, for the space of two thousand

years, aroused in me the suspicion that the truth which it was desired to prove was not contained in the data themselves; that to establish it, the aid of experiment (for example, astronomical observation) would be needed, as in the case of other laws of nature." (1, 92.) Is Bonola right in regarding these words of Lobatschewsky as "opposed to Kant"; or are they not merely *a warning* against taking Kant's stage-properties to be part of his plot?

It may be recalled that Saccheri, among others, had attempted to prove Euclid's "parallel postulate": If a straight line falling on two straights make the sum of the interior angles on the same side less than two right angles, the straights will meet on the side on which the angles are less than two right angles. Assuming that two straight lines can intersect but once, then under the hypothesis that there can exist a quadrilateral with the two lateral sides equal, with two adjacent base angles right angles, and with the remaining two angles obtuse, Saccheri was able to prove the parallel postulate a true proposition. (1, 37.) But this truth implies the falsity of the "hypothesis of the obtuse angle." Wherefore the hypothesis of the obtuse angle implies its own denial (1, 38); i. e., turns out to be a *null proposition.* In trying to exclude the alternative hypothesis (in which "acute angles" replace "obtuse") Saccheri was led to the proposition that a certain pair of straight lines (which we should now call *asymptotic*) must have a common perpendicular at infinity. Since these two straight lines cannot have a common perpendicular at any finite distance, he concluded they could not have one at infinity. (1, 43.) This conclusion (based on a "continuity argument") is illegitimate; such lines may be allowed to have a common perpendicular at infinity, and a consistent system of geometry developed. To propose such a geometry was precisely the innovation of Gauss and of Lobatschewsky. The novelty, however, is but a novelty in definition, and as such cannot contradict Kant's conception of space as the apriori means of individuation; for we may use a set of asymptotic straight lines and the set of their common perpendiculars (the well-known horocycles) as a coordinate system whereby individual points may be distinguished as before.

The development of such a geometry, we have said, is purely definitional and deductive; its judgments are analytic. One may still ask, What then, becomes of Kant's apriori *synthetic* judgments? The logical development of a geometry (at least in terms of classes of points, their inclusion and exclusion) may indeed be made in idea; but must not the possibility of employing such a geometry to "locate" facts of experience depend on what physical measurement shows these facts to be? We answer, the judgment that space is of such and such a geometrical character always involves an apriori synthesis, whose effect is to render geometry experimentally unverifiable, while leaving it objectively applicable. For there always exists a range of choice within which the world is what it is only as the result of a fiat, of a "let there be, and there is." Exactly this condition presents itself to the mind choosing an individuating framework for the data of experiment.

When the hypothesis of the acute angle was found to lead to a self-consistent system, though one in which the sum of the angles of a triangle becomes less than two right angles, it was thought at first that the issue of "truth," as between this hypothesis and Euclid's, could be settled only experimentally. Such was the motive of Lobatschewsky, and Gauss too, for regarding geometry as an experimental science.

In this spirit, Gauss set himself to measure the angles of a triangle whose three vertices lay upon the three mountain peaks, Brochen, Hoher Hagen, and Inselberg (2,233). Within the limits of observational error, he could find no results which indicated that the sum of the angles was less than two right angles. This result however does not yet prove the truth of Euclid, inasmuch as there was here, as in all physical measurement, an experimental error within whose limits lay both geometries. But even had Gauss found the angles measured to sum-up to less than two right angles, the unique truth of Lobatschewsky's geometry would not thereby have been established. For alternative interpretations remain: either (1) we may take the light-rays to be bent, leaving the space through which they pass

Euclidean; or (2) we may leave the light rays straight, interpreting the triangles formed by them in terms of Lobatschewsky's geometry. No compulsion is on us to accept either alternative; no experiment has been devised to establish the one geometry and reject the other. But if experiment is incapable of deciding the issue, then the issue does not have an empirical source and origin. Such a source is recognizable only when the physicist, having *made* his geometry inviolable, proceeds to determine the action of masses, or to decide between the truth of two theories concerning the structure of an atom.

Lobatschewsky, similarly inspired, cherished the thought that the question concerning the existence of asymptotic straight lines might be determined by the parallax of very distant stars, for under the Euclidean hypothesis an infinitely distant star should have zero parallax. (1, 94.) If a line from an infinitely distant point be perpendicular to a base line, all lines from that point will be perpendicular to this base. Under these conditions the angle of parallelism is $\frac{\Pi}{2}$ If, for an indefinitely distant star the parallax should come out positive (the angle of parallelism less than $\frac{\Pi}{2}$) Lobatschewsky's geometry would find empirical confirmation, provided the rays of light were assumed to be straight. But in Lobatschewsky's experiment as in Gauss's, the assumption of the straightness of the rays remains an assumption; the light-ray is taken to define the straight, but experience puts on us no compulsion to accept the definition. To put the previous reflection in another way: either of the two geometries may be condemned, either preserved,—by a suitable choice of definition.

Saccheri and Lobatschewsky developed their geometries under the assumption that no two straight lines can enclose an area, i. e. two straights can meet in only one point. The denial of this assumption and the retaining of the so-called parallel postulate leads to a third geometry; namely, the Riemannian. In this there are no non-intersecting coplanar straights; all straights meet twice, after the manner of great circles on a sphere; all straight, or rather all "straightest," lines are closed,

not open; they are endless, not infinite. This geometry follows from Saccheri's hypothesis of the "obtuse-angled quadrilateral"; which hypothesis implies that two straights intersect more than once. (1, 141.) The two straight lines of his quadrilateral, though making right angles with the base, meet in both directions. If with the diameter of the earth's orbit as base, and as vertex a star more than one-quarter of a (complete) line distant from the base, a triangle be formed, the sum of the two interior angles included between sides and base would be greater than two right angles. It has been thought that here lay the way to an experimental decision as between Riemann and Euclid; but the principles governing interpretation must be the same in this as in the previous issue lying between Lobatschewsky and Euclid. (1) An experimental error attaches to all measurements; and as the divergence of Riemann's from Euclid's geometry depends upon a certain constant (the curvature) which remains Riemannian so long as it is greater than, however close to, zero (zero curvative characterizing Euclid), (7, 84), no experiment can confirm Euclid and refute Riemann. (2) On the other hand, the experiment which might be taken to confirm Riemann and refute Euclid, can do so only after we have *constituted* the path of the light ray our definition of the straight.

Laying aside for a moment the problem of an experimental decision as between geometries, it will be profitable to develop more fully the conceptions of the non-Euclidean geometries. To begin with, Riemann in his *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*, laid it down that portions of a manifold, characterized by the common property of lying within certain limits (and called *quanta*), are to be compared by counting, if the elements are discrete; but if continuous, then by measurement (12, 57). Measurement, meanwhile, consists in a superimposing of the magnitudes to be compared; one magnitude being a measure for the other. But in this method of measurement, it is not necessary that along a given line the scale remain everywhere the same, provided—and this requirement is important—provided a law be found, formulating the change-of-scale.

If such a law can be found, the locality of a point in the manifold may still be uniquely determined, the point individuated from all the rest. The determination of such a scale, or system of scales, characterizes the properties of the manifold, and the task of geometry is to study the properties which an ideal manifold must possess if it would represent the particular manifold, "space."

To analyze out of the properties belonging to quantity-in-general those particular ones which distinguish the manifold of space from other conceivable manifolds, Riemann considered an impossible task. These space-characterizing properties must, he conceived, be discovered through experience and experiment (12, 56): the task for experiment being a search for such facts as are necessary to determine the measure-relations of space. But, for the reasons already set forth, the findings of such a search are never completely determinate: "facts" are always surrounded with a mist of probable error, they are always contingent on sources of constant error arising from unaccounted-for physical influences; their interpretation is always in terms of definitions. Wherefore, "facts" must remain such findings as may be interpreted in the frame of several geometries: within the limits of experimental error, within the range of alternative definitions, there remains a domain of freedom making possible a choice of interpretations fitting the observed data.

Max Born, in an ingenious example, suggests the extremities of interpretation to which the same "facts" may submit themselves (2, 227). Instead of thinking of material objects *as contained in space* (somewhat as objects are contained in a box), one may regard objects *as part of the space manifold itself;* but as parts having a scale of measurement different from that of the empty parts. An object with a certain mass-density would then be identical with so-called empty space, if the object were distended and stretched out sufficiently, the density decreasing with the stretching. As a surface, no matter how uneven and wrinkled, may be flattened out and smoothed by stretching, so the non-homogeneous space of experience would become homogeneous if sufficiently stretched. The bodies or masses are

contractions of space, and require a different scale of measurement in order to be treated geometrically. (4) But to whatever degree space be thought stretched, in the process of smoothing out its wrinkles and indentations its various points retain their individuality with respect to other points. They are still individuated in space coordinates, and the parts of this space are as coordinated as are the parts of Euclidean space. Mass, instead of being foreign to, becomes part of space; and part of the framework of individuation.

A manifold of Riemann's type is of extreme generality, yet if it is to have value it must possess certain invariant properties and obey laws; for the very possibility of a rational experience necessitates that there be a functional law in terms of which the particulars obtain their meaning. The generality of the manifold is offset by a specification of invariants: the number expressing the number of dimensions is such an invariant, and between the variables of two n-dimensional manifolds there may be transformations. Any two points of the manifold may be connected by various arcs; and, if the points are sufficiently near, one of these arcs will be the shortest (an *extremal.*) The important problem is to determine in such a manifold the equations of these particular linear manifolds, called extremals or geodesics.

Before such a problem can be attempted, it is necessary to have an expression for the element of distance between any two near-lying points; and much of Riemann's paper is devoted to finding the expression best fitted to this purpose. As he himself states, there is no absolutely unique expression; several serve equally well. For example, instead of taking the square of this element of distance to be a quadratic homogeneous function of the elements of the variables ($ds^2 = g_{ij}\, dx_i\, dx_j$), the fourth root of a homogeneous expression of the fourth degree, or others still more complicated might be used. The first however is the simplest, and is suggested by the expressions for distance in ordinary coordinate systems.

Riemann's arguments have the interest of presenting a general method of studying a manifold "im kleinem." The possibility

of measurement, says Riemann, requires that magnitudes be independent of their position; for only if the length of a line is independent of its position may it be employed to measure another line (12, 59). Further, we must be able in some way to individuate this length while tracking it about in the manifold. The points of any sub-manifold in the manifold are such a particular set of various combinations of the aggregate of values, $x_1, x_2 \ldots x_n$, as are continuous functions of at least one parameter. The arc is defined by the set of x's determined by all the values of the parameter t in an interval ($t_0 \leq t \leq t_1$). Assuming the x's to be each expressed in terms of a unit, Riemann gave the conditions determining the ds along such a linear continuum. The first of these, after the requirement of continuity, is that the arcs may be partitioned into elements in which the coefficients of the quantities dx_i may be considered constant, at least within a sufficiently small region. In polar coordinates the arc-length, $rd\theta$, depends upon r; but within a small region one may consider the r as constant. The second condition requires that when the element is displaced within the small region along itself the length of the linear element ds remain the same except for infinitesimals of higher order; or, when all the dx's are multiplied by the same constant, the element ds has been multiplied by this constant. To satisfy these conditions the expression for ds must be a homogeneous function of the first degree in the dx's, while the coefficients may be functions of the x's although nearly constant within a small region. Moreover when all of the dx's change sign the sign of ds must remain unaltered. Since this requires a function of even degree, the simplest definition of the expression would make it a homogeneous function of the second degree. Hence ds^2 may be equated to a homogeneous quadratic function in the dx's, and ds equals the square-root of this function. All such functions, $ds^2 = g_{ij}\, dx_i\, dx_j$, as may be transformed one into the other by a functional relation between the variables, belong to the same class; and each class represents a particular geometry, a particular system of measurement, a class of applicable manifolds (7, 63). In any small region a Euclidean or pseudo-Euclidean geometry holds.

While it has been said the region under consideration must be infinitesimally small, it may actually involve all the space within which physics is determined The requirement that the region be bounded and homogeneous is necessary for the relative constancy of the coefficients. The curvature invariants are expressions containing the second derivatives of these coefficients (7, 65). If these coefficients are constant, the curvature is zero and the manifold flat or Euclidean; if the coefficients are not constant, but functions of position, the second derivatives are not necessarily zero and the curvature may be positive, zero, or negative, thus yielding the three classic systems of geometry.

In this way we have sought to sketch-in a remarkable schema by means of which the facts and events of experience may be brought within a determinate system. The schema satisfies all the properties of the categories of quantity. There are the individual units or points marked out by sets of n numbers. There is a plurality of such points, but the plurality forms a totality of interrelated and classified elements. Instead of regarding Lobatschewsky's geometry as contradictory to Euclid's, we see it take its place in a wider schema, within which it represents that particular variation in which the curvature is negative. There are not, it is true, many spaces; "es giebt nur ein Raum," but that "one space" may be conceptually constructed in a number of ways. Riemann primarily extended the range of possible choices. As a schema this frame was developed entirely in an apriori way, i. e., definitionally or analytically. The synthetic constructive feature appears only when such an array is worked into physics; then the more or less conceptual schema becomes intuitive, a means for moulding phenomena so that they may be brought under control and within the range of predictability; to which end, some one such schema is a necessary means.

In contrast with the rigid schema of Euclid, this Riemannian manifold is plastic, of an infinite flexibility. For the curvature may vary throughout the continuum in which Euclid's system is but a point. The bringing down of this—in itself rather abstract —theory from the shelves of mathematics to the problems of experimental physics was first accomplished through Einstein's

general theory of relativity. This theory, however, forms a later chapter.

Riemann's conceptions were taken up by Helmholtz, who frequently discussed the origin of the geometrical axioms and usually defended a naïvely empirical point of view. His argument takes the familiar form: since Euclid's are not the only axioms conceivable, they are not necessarily found in experience (8a, 55). For Helmholtz, the facts on which geometry rests are the motions of rigid bodies. In finding the length of an interval, the measuring rod must be applied repeatedly, and in order that the readings have value, the rod must be not only mobile but also as rigid and unalterable as possible. Helmholtz believed the axioms to be concerned not with space relations simply but also with the mechanical deportment of solid bodies in motion; and our knowledge of such bodies is derived from our experience of their behavior. Rigid bodies are elastic solids of great resistance (8a, 66); if no force affects them they retain the same form in every part of space. Belief in the existence of such bodies has no more certainty than other empirical knowledge, and the laws based on it are but inductions.

Geometry then rests on certain unproved propositions, e. g., the axiom, that there exists but one straight line between two points; yet to this very axiom pertains no conceptual necessity, since in Riemannian manifolds we have seen an infinity of straightest lines allowed to connect conjugate points. Similarly, there pertains no necessity to the parallel postulate, since in Lobatschewkian manifolds it is allowed not to hold. What is the source of such propositions which seem true and yet are incapable of demonstration? Philosophers say they have a quasi-divine source in the human understanding; but, Helmholtz would ask, may not the results of everyday experience be all too easily mistaken for necessities of thought? Our familiar dealings with objects seem to show that through a point exterior to a line only one line may be drawn parallel to the given line (8a, 67). The facts of experience encourage an habitual expectation; this engenders a prejudice; which finally becomes a "necessity

of thought" (8a, 45). Kant claimed for his apriori synthetic judgments a compelling authority not enjoyed by inductions from experimental data. But if in thought such apriori synthetic axioms may be replaced by others, then they enjoy no such necessity, and the determination which of all possible sets of axioms is the *true* system becomes an experimental task.

This rough-and-ready empiricism of Helmholtz was subtly tempered by Poincaré, for whom experiment cannot refute geometries, yet may guide us in the selection of a schema best adapted to the needs of physics. The motives of Poincaré may be introduced in the form of certain critical reflections on Helmholtz.

Helmholtz's development of the form of the element of distance (8b) is much more restricted than Riemann's, and would never be adequate for a foundation of metrical geometry. Relativity has come to show how bodies Helmholtz would have accepted as rigid may when moved undergo contractions; and whereas Riemann gave to his ds a form interpretable even under these conditions, Helmholtz's method leaves entirely undecided the nature of ds in such cases. For any transformation of a general manifold, the invariant and covariant character of the ds of Riemann harmonizes better with the conditions apriori of an individuating equipment; for with respect to all the transformations the ds represents something unique and individual in the manifold. However general, the conceptions of Riemann are not haphazard but obey defined laws; whereas Helmholtz arbitrarily seizes on a " rigid length " as on something *given* in experience (8a, 50). Such an *immediate given*, however, no more exists than do two equal durations: the most accurate chronometer must occasionally be corrected to bring it in accord with sidereal time, as given by the rotation of the earth relative to the fixed stars. But the rotation of the earth itself may be gradually slowing down, the intervals of daily rotation becoming longer and longer. In presence of such considerations, Poincaré would suggest that we introduce a convention *and call these intervals equal.* Instead of equal durations and rigid lengths being given

as data, they are conventional abstractions or ideals. Consequently Riemann's apriori treatment of a manifold, with its characterization by means of curvature (constant, if a portion of the manifold is to be mobile without distortion) brings out more satisfactorily than does Helmholtz's empiricism the true relation of geometry to experience.

That the rotation of the earth is slowing down, astronomers deduce, as Poincaré points out, from the law of the conservation of energy; for the heat-energy developed by the friction of the tides can be offset only by a diminished kinetic energy of rotation. Again the secular acceleration of the moon, calculated according to Newton's law, would be less than that established by observation, were not correction allowed for retardation of the terrestrial rotation. The astronomer's time thus involves Newton's law: the astronomer *defines* duration in such a way that Newton's law and the principle of the conservation of energy be verified (11, 227). Since Newton's law is an induction and an approximation, this definition of duration has no absolute value. The motives for such a definition may be summed up thus: "Time should be so defined that the equations of mechanics may be as simple as possible. In other words, there is no one way of measuring time more true than another; that which is generally adopted is only more convenient." (11, 228.) (How true this statement, first becomes clear through Einstein's special and general theory of relativity, in which we find many a preconceived idea about space and time modified in order to give to the equations of mechanics a simplicity retained for all possible coordinate systems.)

Just as we have no direct intuition of the equality of durations nor of absolute rigidity, so have we none of simultaneity: "The simultaneity of two events or the order of their succession, the equality of two durations, are to be so defined that the enunciation of the natural laws may be as simple as possible. In other words all these rules, all these definitions are only the fruit of an unconscious opportunism", (11, 234). Like all mathematical concepts, "simultaneity" is matter of definition, of apriori analytic judgment; whereas, such is the relation of these concepts

to experiment, that by the time we are ready to assert space and time to exist as defined, we have already put into the world of experience the space-time conditions we find there,—an apriori synthetic judgment has played its part in the resultant " finding. "

Guided by such considerations, Poincaré might well conclude that axioms are neither apriori synthetic, nor aposteriori synthetic; but, disguised as implicit definitions, they are apriori analytic judgments. To ask then whether Euclid is true or false, is no better inspired than to ask whether the metric system is true and older measures false (11, 65). Yet this interpretation leaves much to be desired; for those "disguised definitions" are surely not meant to remain nominal, but to be given an objective reference. Definitions have little value unless their objects have empirical existence: it is precisely its superior applicability that gives the ds of relativity its advantage over Riemann's ds. Any system of postulates characterizing a manifold is indeed but a set of implicit definitions of the entities of that manifold; but when the manifold becomes the space-time continuum of nature, when the theorems are employed in the business of physical prediction, then a synthetic aspect appears. For then all propositions must assume, if only tacitly, that the space-time of the experienced world actually possesses the properties piled up by the system of implicit definitions. But if our pursuit of the historic dialectic leads to the conclusion that physical experiment is no such process as will enable us to find the "space" of one geometry present, that of another absent, and if nevertheless physical events must take place in space, it follows that the space (or space-time) in which they take place must be *imposed* before the finding process begins. If the theory of relativity makes clearer the sense in which this imposing of a space-time frame is possible, and yet not so final as to be beyond reconsideration, it will have done much to support the Critical contention. To say, " a certain space-time *exists* in experience," is surely a synthetic judgment; to add, " because we put it there, " reveals the apriori grounds for this synthetic judgment; to reflect (which is the modern extension of the Critical doctrine), "and we may be advised by our experience of things in this space-time setting,

to change the setting and begin experimenting anew"—this may be a more critical as well as a more modern interpretation of Critical Philosophy.

III

The Restricted Theory of Relativity

Space, Time, Kinematics

The absolute space of Newton having become identified with the aether, the question arose whether an aether wind does not result from the motion of the earth through this aether. Michelson and Morley undertook to determine experimentally whether any difference could be detected between the times taken by a light ray to travel to and fro over two equal intervals, the one taken parallel, the other perpendicular, to the direction of the earth's motion. Such a difference should appear if an aether wind exists, but none was found; although Michelson was certain he could have detected the 100th part of the predicted time difference, had it existed (2, 165). The inference follows that to an observer moving with reference to its source, the velocity of light is constant in all directions. Furthermore, de Sitter, observing the velocity of light coming from certain binary stars, found this velocity unchanged whether the source were moving toward or away from the terrestrial observer (16, 24).

Before Einstein set up a new definition of simultaneity, special ad hoc hypotheses had been invented to account for this zero difference in time. Fitz Gerald considered it sufficient to allow a body to contract by an amount dependent on its velocity in the direction of its motion (2, 165). But Lorentz discovered that beside this contraction another postulate was needed; namely, such definition of a local time for each reference system as would leave the electromagnetic equations unaltered in form (covariant) when referred to any other reference systems in uniform motion of translation relative to any given one.

In this "electrodynamic theory" the facts are taken care of, but the notions of contraction and of local time affect us as arbitrary. To avoid this effect, a new interpretation of the foundations of physics was attempted, adjusting the description of space and time to the phenomena of bodies, rather than modifying the laws of bodies to fit traditional conceptions of space and time. Might not "contraction of moving bodies" and "local time" be eliminated by reforming the "presuppositions of experiment"? May not space and time be interrelated in one continuous four-dimensional manifold (space-time) in which "events" may still be located as taking place somewhere at a certain time, though the "frame" by which this individuation is effected may be of very different pattern from any imagined by Euclid, Newton, Kant? If so, would the new frame be less apriori, its application less a matter of synthetic judgment than the old?

The motives for such a change of individuating frame are not difficult to set forth. Mechanics in its Galilean form does not obey the principle of covariance; electrodynamics in the Lorentz form does; yet both may be made to fall under the same description, provided our account of both be based on a new definition of the simultaneity of events. This is the new development of an ancient and obvious principle of "relativity of motion"; the principle, namely, that of two moving bodies either may be considered at rest, that consequently no absolute reference frame, independent of choice, can be "found." A frame, to be used, must first be *made*, then *imposed*.

Such, it would seem to the writer, are the general considerations which best serve one in relating a new development to old motives of thought; any adequate test of their justice must examine in greater detail the new theory they are meant to announce.

Turning to the work of Einstein, one finds the scientist building the physical world anew on two postulates: (1) relativity, and (2) constancy of light-velocity. Part of the work of relativity has been to define motion in terms of moving coordinates. For

example, the Galilean transformation, $x' = x - vt$ (v being velocity, and t, time) defines a continuous translation of the coordinate frame x′ with respect to x. Through the constancy of the velocity of light, it is possible to introduce time as a fourth dimension; for the very simple consideration that any distance equals the velocity of light multiplied into the time it takes to travel the distance (i. e., $x^2 + y^2 + z^2 = c^2 t^2$) gives a hint that an interval squared in this manifold may be given the form, $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$; and that for the path of a light ray $ds = o$. Space-time thus becomes a unified equipment for the individuation of point-events.

The relation between space and time rests on the conception of "simultaneity" of mutually remote point-events. Poincaré had already remarked that as in view of the slowing-down of the earth's rotation there is no absolute equality of time intervals, so there is no absolute simultaneity of events: time intervals are called equal by convention, and this convention is to be so framed as to render the equations of mechanics as simple as possible. Relativity takes the step of defining what is meant by the simultaneity of events separated by a distance: If at a point midway between two given points a spark occurs, the moments at which the flash is seen at the end points are *simultaneous moments*.

Consider the points A, C, B (C the midpoint), and a second system of equal intervals, A′, C′, B′, moving relatively to the first. At the moment C and C′ coincide let a flash be emitted at the midpoint. If A′ moves toward C, to an observer in A, C, B, the arrival of the flash at A′ appears prior to its arrival at A, since A′ moves to meet the flash; for like reason, B′ seems to receive the flash later than does B. But in A′, C′, B′, the moments of the arrival of the flash at A′ and B′ also satisfy the definition for simultaneity. Two events therefore which are simultaneous in A′, C′, B′, are not simultaneous when observed from A, C, B (2, 173). There seems to result a contradiction; but all here contradicted is the Newtonian assumption that *simultaneity* defined for one system of space-coordinates will be defined for all systems, independently of their relative motion

or rest. To avoid this assumption, we may proceed as in the relativity theory:

Let the coordinate-system, x′, y′, z′, be parallel with the system, x, y, z, and let its motion be one of translation along the x-axis with velocity v; let the moment of coincidence of the origin be $t = t' = o$. A duration, $\triangle t'$, and a length, $\triangle x'$, are to be expressed in terms of $\triangle t$ and $\triangle x$ respectively.

In the system S′, suppose a light-signal to be sent perpendicular to the x′ axis to a point M at a distance h, and thence reflected to the source. The source in the meantime will have moved, so that the ray will have traversed two sides, P M and M Q, of an isosceles triangle in the system S. Letting c represent the velocity of light, the time taken for departure and return in system S′ is $\triangle t' = \frac{2h}{c}$. But for an observer in S, $\frac{1}{2}$ PQ: PM = v: c. Also, PM: h = PM: $\sqrt{(PM)^2-(\frac{1}{2}PQ)^2} = 1:\sqrt{1-\frac{(\frac{1}{2}PQ)^2}{(PM)^2}}$ $= 1:\sqrt{1-(\frac{v}{c})^2}$. Introducing for compactness the symbol α, we may write, PM = h: $\sqrt{1-(\frac{v}{c})^2} = h\ \alpha$. But since $\triangle t = \frac{2\ PM}{c} = \frac{2h}{c}\ \alpha$, we have

$$\triangle t' = \frac{\triangle t}{\alpha}; \quad \triangle t = \alpha\ \triangle t'$$

This equation expresses the factor by which time-intervals ($\triangle t'$) of the moving system are "dilated."

Now imagine a signal sent along the x′ axis from P to a mirror N, distant $\triangle x'$ from P. The time during which the light travels to N and back is $\triangle t' = \frac{2\triangle x'}{c}$. For an observer in S, the mirror when the light reaches it will have advanced to M; and when the light returns to P, P will have moved to Q. Let the distance PN as measured in S be $\triangle x$. Let the time of departure be $\triangle t_1$ and of return $\triangle t_2$. Then $P\ M = c\ \triangle\ t_1 = \triangle x + v\ \triangle t_1$ and $Q\ M = c\ \triangle t_2 = \triangle x - v\ \triangle t_2$. Hence the time of journey as measured in S is $\triangle t_1 + \triangle t_2 = \triangle t = \frac{\triangle x}{c-v} + \frac{\triangle x}{c+v} = \frac{2\triangle x}{c\,(1-\frac{v^2}{c^2})} = 2\,\alpha^2\,\frac{\triangle x}{c}$.

But since $\triangle t = a\ \triangle t' = a\ \frac{2\triangle x'}{c}$, we have

$$\triangle x' = a\ \triangle x; \quad \triangle x = \frac{\triangle x'}{a}$$

This equation expresses the factor by which lengths ($\triangle x'$) of the moving system are "contracted" (the Fitz Gerald contraction).

The event distant x from the plane y, z is distant $x - vt$ from the plane $y' z'$. We have, $\triangle x' = x'$; $\triangle x = x - vt$, whence, since $\triangle x' = a\ \triangle x$,

$$x' = a\,(x - vt) \quad (1)$$

Two events which are simultaneous in one system are not simultaneous in the other. Eddington has given a simple calculation of the difference (5, 28). When a signal is sent from P (at a time t_1) to N, and returns to P (at a time t_2), the time in S' of arriving at N is $\frac{t_1 + t_2}{2}$. For S, the duration of the outgoing is $\frac{\triangle x}{c - v} = \frac{a^2\ \triangle x}{c^2}\,(c + v)$; and of the returning, $\frac{\triangle x}{c + v} = \frac{a^2\ \triangle x}{c^2}$ $(c - v)$. The instant of arrival at N must be taken as $\frac{a^2 v\ \triangle x}{c^2}$, half the difference, later than the half-way instant. But as $\triangle x' = a\ \triangle x$, the departure from simultaneity is $a\ \frac{v}{c^2}\ \triangle x'$. Addition of this departure to $\triangle t = a\ \triangle t'$, a dilated time lapse gives the time in S between two remote, non-simultaneous events in S'. That is, $t = a\,(t' + \frac{v}{c^2} x')$

Solving this equation and (1) for t' by elimination of x' we obtain

$$t' = a(t - \frac{v}{c^2} x) \quad (2)$$

Equations (1) and (2) constitute the Lorentz transformations.

It will be noticed that as $c \doteq \infty$, (1) and (2) approach the

forms $x' = x - vt$; $t' = t$; the Galileo transformations which leave covariant the expression for force, $f = m\,a$.

Again, these Lorentz transformation correlating "inertial" systems in uniform translation with velocity v, leave covariant the interval $ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2 = dx'^2 + dy'^2 + dz'^2 - c^2\,dt'^2$.

By solving for x and t in terms of x′ and t′ one finds only a change of sign for v, an indication of the symmetry of the principle of relativity. It is entirely an apriori principle set up by an act of choice. Cassirer contrasts this formal principle with the material principle of the constancy of light velocity (3, 378). The notion of curves of zero length, ds = o, has its origin, however, in the minimal lines of differential geometry which have here found practical application.

With the values 1, 1, 1, $-c^2$, for the g_{ii}'s, and the value zero for the g_{ij} $(i \neq j)$'s, the form of ds is brought in alignment with the structure of a manifold which even in its generalized form retains (by the principle of equivalence) the restricted theory for its elements.

Frequently the interval ds, as an element of the world-line of a particle, receives the designation "local time"; because, along the world-line t′ being defined by $x = vt$, $dt' = dt\left(1 - \frac{v^2}{c^2}\right) a = \frac{dt}{a}$; or $c^2\,dt'^2 = ds^2 = c^2\,dt^2 - dx^2$. Beside this *time-like* ds, there is a *space-like* ds. For, when $dt' = dt - \frac{v}{c^2}dx = o$, $ds = dx' = dx\left(1 - \frac{v^2}{c^2}\right) a$; and $ds^2 = dx^2 - c^2dt^2$. If $ds^2 = (dx - c\,dt)(dx + cdt) = o$, $x = ct + k$ gives the world lines for light rays; $(x - ct)(x + ct) = T_1 \times T_2 = \pm 1$ defines a pair of hyperbolas which mark off the unit distance, $x = \pm 1$, $t = \pm {}^1/_c$, (2, 180.)

An interesting possibility presents itself in that two events which have a certain sequence, say A before B, in one system may in another system have the reverse order, B before A. Suppose $t_2 - t_1 < o$, where t_2 and t_1 are the times of two events referred to a rest-system. Now, provided the interval between

the two events be space-like, a system may be found moving with a velocity such that the time of occurrence of these two events seems to take place in the reverse order: $t_2' - t_1' = \alpha \left[t_2 - t_1 - \frac{v}{c^2} (x_2 - x_1) \right] > 0$, if $x_2 - x_1 < 0$, and if v is sufficiently large. (15, 28.) The relation *before-and-after* evidently has lost its absolute sense; but this insight could only be revealed by the magnifying power of analysis, for it is beyond direct experiment.

These novel considerations concerning the relativity of (1) simultaneity, (2) length, (3) duration, and (4) before-and-after, have been presented as elaboration of a pseudo-Euclidean type of manifold which nevertheless does not conflict with the notion that space-time, as an apriori intuition, forms an individuating equipment making experience possible.

While the space-time or metrical continuum has been depicted as an apriori development, it must not be thought that it is without bearing on problems of experimentation; on the contrary it helps to further the experimental method. The objection against absolute space, time, and simultaneity was that they were empty abstractions beyond the reach of experiment. The point events of the continuum have not only a conceptual being but an immediate observational reference. Furthermore, it may be emphasized that the reduction of probable error does not really take place by simple refinement of the measurements but indirectly, by the construction of a more inclusive system of explanation within which many independent "inductions," to be included as deductions, must be treated as approximations. In this sense, relativity has rewritten many an equation by introducing a second-order term; it has so treated the formulas

expressing the Doppler effect; the aberration of light; the Fresnel convection formula.* (2); (16, 55).

A marked achievement of relativity has been its derivation of many equations in more precise form eliminating all special ad hoc hypotheses, all the result of a reformulation of the presupposition of physics, the space time individuating equipment. Experimentation alone could not have given this insight; it alone can verify what has been predicted by analysis. But the method of relativity in redefining the presuppositions of experiment makes it an instrument of penetration (3, 378); it *formulates the more penetrating question*, probing behind appearances toward that which appears—that is to say, toward *the more real.* The real length was defined as the mean with zero probable error. This is never attainable; but the theory of relativity enables science to approach the goal; puts the questions to answer which experimental method must refine its measurements.

IV

Relativity Dynamics

Matter and Causality

Analysis, geometry, and kinematics are sciences which have the task of establishing propositions whose truth depends upon the meaning of their terms, whether numbers, points, instants,

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* The Fresnel formula, giving the velocity of light in a moving medium (e. g., water) was obtained on the assumption that the medium carries the aether along, partially. The formula may however be derived from the equation for the addition of velocities $\frac{x'}{t'} = \frac{x - vt}{t - \frac{vx}{c^2}}$ or $v' = \frac{v_1 - v}{1 - \frac{v_1 v}{c^2}}$ where v_1 is the velocity of a particle in the coordinates [x t'] and v' its velocity in [x' t']. If v_1 is the velocity of light with reference to the moving water, v is the velocity of the water, then v' is the velocity of the light with reference to the observer (16, 60). The Fresnel formula is obtained thereby in a stricter form, since it involves a second order term $\frac{v_1}{c^2}$ to detect which necessitates refinement of experimental technique.

or events. In that the meaning of terms is here in play, these sciences are to be contrasted with logic; but in so far as they confine themselves to meanings, they do not appeal to experiment. In the "physical" sciences built on these, the introduction of the *mass* and *energy*, concepts derived from experience, immediately imposes the empirical check of measurement; and the truth of the equations involving such terms rests on experiment. The velocity of light, the velocity of propagation of electromagnetic energy, cannot avoid a material reference; and to this extent relativity-kinematics is subject to experimental verification. But insofar as coordinates are concerned, choice is free, or only determined by convenience; they may be chosen Galilean, or curvilinear, or of any other self-consistent type.

Mass reveals itself in the interaction of bodies coming into collision, and therewith undergoing changes in velocity. When two "perfectly elastic" spheres collide the ratio of the accelerations they impart to each other is taken to be constant; an assumption on which Mach based his well-known definition of *mass*. (10) Of two colliding bodies, let a, be the acceleration the "unit body" imparts a_0, the acceleration it receives in return.

Then, $-\frac{a_0}{a_1} = k_1$ where the minus sign preserves the opposing senses of a_0 and a_1. Similarly, for another body $-\frac{a_0'}{a_2'} = k_2$

These laws are true within the limits of experimental error. The inert mass of the first body is defined as $k_1 = m_1$ and that of the second, as $k_2 = m_2$. These equations are equivalent to Newton's first two laws.

Furthermore when these last two bodies collide $-\frac{\bar{a}_1}{\bar{a}_2} = k$

and the question suggests itself whether $\frac{m_2}{m_1} = k$. The third law of the equality of action and reaction $\bar{a}_1 m_1 + \bar{a}_2 m_2 = 0$

answers the question, for $\frac{m_2}{m_1} = -\frac{\bar{a}_1}{\bar{a}_2}$ (By integration the third law becomes the law of the conservation of momenta.)

In this form Newton's laws of motion are equivalent to a definition of mass. An empiricist believes that because these laws depend upon experiment for their truth, science has its origin in experience. Science in setting up these laws has, nevertheless, tacitly presupposed, in addition to the individuating equipment, the apriori synthetic judgment: experience is possible only through the notion of a necessary connection of phenomena; the task science performs is to find the precise law for the connection, and this law is always subject to revision; the moment science begins to experiment it assumes law to exist.

When Kant was considering the "laws" which the investigator's "understanding" imposed upon whatever results his experiment might yield, he included three of a markedly Newtonian flavor:

(1) Throughout any physical transformation the quantity of something (= "substance") remains the same.
(2) All changes in phenomena follow the law of cause and effect.
(3) All bodies are in perpetual interaction.

These principles do not intend to formulate laws subject to experimental revision, but state conditions without which the formulation of such laws is impossible. They are true apriori synthetic judgments.

(1) Of these three, the first is consistent with Lavoisier's law stating the constancy of the quantity of mass during physico-chemical transformation. If Kant's principle is taken to be identical with that of the "conservation of mass," it would seem to require modification, since in relativity physics mass increases with velocity. If, for example, two elastic bodies move along the x axis, one with velocity u, the other with velocity $-u$, for an observer moving past them with velocity v, they have velocities $u_1' = \frac{u - v}{1 - \frac{u v}{c^2}} u_2 = \frac{-u - v}{1 + \frac{u v}{c^2}}$ (16, 67). Let the masses be m_1 and

m_2, which then at the moment of collision move past the observer with velocity $-v$. If mass is conserved during collision, the sum, $m_1 + m_2$ remains constant; and if the law of the conservation of momenta is valid, $-(m_1 + m_2)\, v = m_1 u_1 + m_2 u_2$.

By substitution $\frac{m_1}{m_2} = \frac{1 - \frac{uv}{c^2}}{1 + \frac{uv}{c^2}} = \left[\frac{1 - \frac{u_2^2}{c^2}}{1 - \frac{u_1^2}{c^2}}\right]^{1/2}$. If for the case $u_2 = 0$, we write $m_2 = m_0$ (the mass of each of the two bodies at rest), then $m_1 = \frac{m_0}{\sqrt{1 - \frac{u_1^2}{c^2}}}$ The mass therefore increases with velocity.

Here, however, the mass of the *two bodies* and their momenta are conserved during impact (16, 68); and it will be remembered that the most restricted interpretation of Kant's argument for the "conservation of the quantity-of-something" could not represent him as requiring the constancy of "masses," but only of "mass." Now, by expansion the expression for m, may be given the form, $m = m_0 + \frac{1}{2} m_0 \frac{u_1^2}{c^2}$; $m - m_0 = \Delta m = \frac{E}{c^2}$ where E is kinetic energy (5, 32). According to this equation, 9×10^{20} ergs = 1 gram; kinetic energy has the property of inertia, or at least kinetic energy and inertial mass are proportional (16, 41). This relation has been found to exist no less between electromagnetic energy and mass (16, 180). The light-ray, a form of energy, has momentum and exerts pressure on a recipient surface. Wherewith, the problem of the conservation of mass becomes part of the more general problem of the conservation of energy.

The "energy" whose conservation is here in question is a total made up of many "kinds of energy" (kinetic, potential, thermal, gravitational, electro-magnetic and chemical), and these undergo perpetual transformation one into the other. The

expression for this total, the theory of relativity leaves invariant. Kant himself could have asked no more.

But the ultimate principle which Kant expressed in the language of his day is probably to be deduced from considerations deeper-lying than any affected by the theory of relativity. Poincaré had already noted that all equations stating the transformation of one kind into another are differential equations of the first order. Hence when all the terms are equated to zero and integrated, something remains constant (constant of integration) (11, 118). The notions of variation and constancy are to this extent inseparable; and a profounder revolution than any "relativity" has dreamed of, would be required to represent nature as mathematically describable, yet in such wise that no constant ("substance") emerged from the description.

(2) Some form of Kant's second "Grundsatz," concerning causality, science accepts as its ideal the moment it begins to seek a law. Relativity has furthered a certain mechanical ideal which seems accurately to express Kant's meaning: To develop equations by which, given the values of the dependent variables for one value of the independent variable (t) the dependent values may be calculated for all values of t (= time). In its pursuit of this ideal, science finds its principal embarrassment in the "indeterminateness" resulting from various "probable errors." The second order factor introduced in all relativity formulas suggests refinement of experiment with a consequent reduction in the size of probable error; and where, as in the variation of mass with velocity, the actual quantities involved are generally too small for experimental detection, relativity has simplified description *without loss of power of prediction.*

(3) The postulate of the mechanical ideal embraces beside the "law" of causality the apriori synthetic judgment that all bodies are in perpetual interaction, meaning that they move according to some law of attraction, of repulsion or of both. This judgment raises the two questions: (a) whether it is possible to build up a physics without some law of interaction, (a statistical mechanics at first thought seems possible without one); and (b) whether a physics can develop without presupposing some law of instantaneous influence.

(a) It has been plausibly argued that since all substances reveal cohesive forces these presuppose some kind of interaction between the particles. Moreover at the basis of the formulation of the mechanical ideal lies the n-body problem and without some law of interaction this problem remains indeterminate. Without such a law it is impossible to individuate particles, as necessary in the mechanical ideal, for so far as statistics are concerned two distributions in which particles of equal masses are interchanged are entirely equivalent.

(b) Before relativity's advent such laws as Newton's inverse-square law and the inverse square-law of electrostatics were action-at-a-distance laws. The former furthermore is a law of instantaneous influence whereas electrostatic influences are propagated. The law of Boscovitch and Kant attempting to combine gravitation and elasticity by letting bodies attract according to the inverse square-law and repel according to the inverse cube-law is likewise a law of instantaneous action at a distance. The mechanical postulate states that from a given distribution at one moment all past and future distributions may be calculated, and requires that the influence of each particle be imparted to all others immediately.

Relativity both dismisses these classic arguments and confirms the apriori conditions they sought to establish: it interprets all such instantaneous action at a distance as part of the geometrical structure. As, however, the motions involved are no longer straight, the discussion passes to the General Theory.

V

The General Theory of Relativity

Ateleological Law

The special or restricted theory of relativity introduced systems of coordinates in mutual motion, but this motion was confined to the rectilinear and uniform. In the general theory

this restriction is set aside, and systems of coordinates accelerated with respect to each other are studied. These reference systems are to retain equivalent status; no one may be be regarded as uniquely at rest: a "falling" stone may be thought of as stationary, and the earth moving toward it; the "rotating" earth may be thought of as still, the rest of the universe revolving around it.

The step, however, is an audacious one, for Newton had weighty reasons for introducing the conception of an absolute space. In order to gain the necessary insight, not only must we abandon the notion that between the sun and planets are "traces" holding them together, but also the notion that bodies subjected to no force move in straight lines. In the four-dimensional space-time continuum of relativity, each moving particle follows a "straightest" but not necessarily a "straight" path.

The demonstration of this new view of natural motions, must begin by establishing the proportionality, if not the equivalence, of inertial and gravitational mass. It is an observed fact that a body experiences in a gravitational field a force $\frac{d}{dt}$ (m v) which for different bodies at the same place varies with the property called "gravitational mass" (M). Such a force is the "weight" indicated by ordinary scales. Any two bodies have the same weight when on being placed in the two pans of a balance, they do not disturb its equilibrium. If two such bodies found to be equally heavy are placed in one pan of the balance, and in the other pan another body such that the equilibrium is again undisturbed, this new body has twice the weight of either of the other two. By this method a scale of weights may be set up, as by letting bodies collide a scale of inert masses may be established. The question naturally suggests itself whether between these two scales in which the same bodies may be arranged, any relation can be established. Now, it is an accepted fact that in a vacuum two bodies of different weight fall with the same acceleration. The force, $\frac{d}{dt}$ (mv) varies with the gravitational mass M. Postu-

late $\frac{d}{dt}(m v) \propto M$, or $m a = g M$, where g is a factor of proportionality, the same for all bodies at the same place and referred to the same central mass (e. g. the earth). Consequently the ratio of the forces which two bodies undergo is $\frac{f_1}{f_0} = \frac{M_1}{M_0}$. If M_0 is defined as the mass of a body of unit mass, then M_1 is the mass of the second body.

Furthermore it is known that while g does not vary with m, it does vary with position in the field; in particular, with radial distance from the central mass. Wherefore g may be called "the intensity of the gravitational field," defined by the equation, $g = \frac{ma}{M}$ where m is the inertial mass; a, the acceleration due to gravity of a body at the place; and M, the gravitational mass. According to the experiments of Eötvös (17,525) M varies with m, or $M = \kappa m$. The units of M and m may be so chosen that $\kappa = 1$; whereupon $g = a$, the acceleration due to gravity at any point of the field.

As the result of this development, inert mass and gravitational mass are numerically equal. They are the same for the same body. This numerical equivalence makes possible the amalgamation of the phenomena of inertia and of gravitation; and it is but a step to consider a gravitational field as a continuum of such curvature that the natural path of a body is the path along an extremal in this continuum. To oppose to the gravitational force acting on a planet, say, a counterbalancing centrifugal force just sufficient in magnitude to keep the planet in its orbit is no longer necessary. The specific dimensions of mechanics becomes *mass* in the one form, "inert mass."

Here as frequently an important theory seems to be based on an empirical fact; in this case, the proportionality of M and m. In many situations of the sort nature impresses one as having been more rational than the physicist. In classical mechanics this law was not reckoned among the fundamental principles but was rather accepted "as an accidental gift of nature." In the

words of Born: "A formulation of the law of motion is now to be found in which inertia and gravitation amalgamate to a conception of higher order; and in such a way that the motion [of a given mass] is determined only by the distribution of the remaining masses in the universe." (2, 22.) By eliminating all forces which are exactly balanced by counteracting forces, the principle of observability has been carried through; namely, the principle which would confine the data of science to observable facts. Fictitious forces, since unobservable, should be eliminated as non-existent. But what here appears to be an empirical datum, becomes through insight a motivating principle for the general theory of relativity.

This principle makes possible an extension of kinematics to let it include accelerated and rotating systems of coordinates. What is called centrifugal force becomes the gradient of a potential of centrifugal force, and both this potential and the gravitational potential are interpreted as expressing properties of the geometrical manifold. A potential function expresses the metrical properties of the coordinates; the notion of attractive forces has become meaningless; or, if such entities as attractive forces are assumed, they cannot be distinguished from fictitious geometrical forces. Thus "action at a distance" finds its long-sought explanation in its sudden disappearance: the invention of such reference-frames as relativity proposes, permits a purely *kinematical* imaging of the flux of facts called "mechanical." Planets move about the sun, no longer held to their orbits by a mysterious force, but in undisturbed motion along geodesic lines; any body in a gravitating system may be considered stationary; what point is to be regarded as the origin depends entirely on apriori choice, and that choice on convenience alone.

The postulate of the equivalence of gravitational and inertial mass expresses the notion that in its elements a gravitational field (that is, a field in which all bodies undergo the same acceleration) may be produced artificially by an accelerated coordinate system. By the same method, a small portion of the earth's

gravitational field may be transformed away, as in a freely falling elevator weight has been so transformed.

The question must then arise whether *all* gravitational effects may not be transformed away by an appropriate system of reference. But this possibility does not exist for the *total* field surrounding a central mass. In order to retain the principle that the motion of any body exclusively under the influence of gravitating masses be interpreted as inertial motion, Euclidean geometry must be abandoned. This principle is an outcome of a thought emphasized by Clifford (4); that, namely, the metrical properties of space are wholly determined by the masses of bodies. This, as opposed to the classical deduction of mass and inertia from a reciprocal action between bodies, an action bringing them together when apart, or thrusting them apart after collision. In this way a synthesis of all mass-dynamics may be attained; but (as Born points out) "the theory that the inertial motion relative to the inertial system is uniform and rectilinear, i. e., the ordinary law of inertia, now comes to an end." (2, 223.)

With this law passes the unique position of Euclidean geometry, for Euclidean geometry as physical geometry rests on the straight lines of inertial motion. And as in this geometry the straight line, circle, plane, and so on are absolutely rigid entities, to give up this absolute rigidity, to require of one the conception that straight from another point of view is curved and curved is straight, seem to imply that everything is tottering. It seems as if the ground beneath us were giving way, yet such a catastrophe must be risked. An object moving in a straight line over a rotating reference frame will appear to an observer in the rotating system to move along a curve. (2, 42.) In any small region space is approximately homogeneous just as the neighborhood about any ordinary point on a surface is approximately a plane. Consequently there exists a coordinate system in which the point-event interval obeys the Lorentz transformation. In any general coordinate system the interval has the general quadratic form.

All motion, according to the general theory, becomes a question of coordinate systems changing their position and dimensions

with the independent variable, time. Reference systems do not exist in nature nor in things, they are entirely the creation of the human mind and have their origin in the recognizing (or classifying) and individuating equipment, the minimum presupposition to the meaning of experience. Yet the seeking and creating of such an apparatus, with the purpose of retaining the identity of each point event in all reference systems, is but an extension of the familiar *principium individuationis*, by which the Law recognizes an established alibi as a final defense. Even if two persons are indistinguishable on the basis of any system of physical measurement, they remain two if their lives form self-identical but mutually distinct world-lines in the four dimensional manifold of space-time. The whole value of the general theory of relativity lies in its greater ease of predicting mechanical events. The general theory of relativity turned up the three new cases of ateleological prediction not suggested by previous theory.

Instead of conceiving space, as did Euclid, everywhere homogeneous and plane, Einstein thinks of it, as does Riemann, in analogy with a surface on which may be constructed a meshwork of geodesics. As a particle subjected to no force moves along a straight line, so a particle restrained to a surface will move along a geodesic of that surface. It is in this wise a planet moves through its orbit, for in a positively curved continuum, like that surrounding a point-mass, the shortest of straightest paths are closed. A rotating mass is in a certain relation with other masses; and only the reciprocal relation has importance: it alone enters the definitional structure to account for "centrifugal effects": the revolving stars flatten the earth at the poles. There exists no particular reference frame in which bodies are thought to move, for each body is a moving reference frame unto itself and the general principle of relativity states that the laws of physics must be expressible in equations which are of exactly the same form in whichever system they are expressed.

In one sense this requirement of "covariance" sets up a definition of the nature a principle of physics must have, in order to be called a law. While any given law (e. g., that which makes

the force components acting on a particle equal to the negative derivatives of a potential function in the component directions) may not satisfy the principle of covariance, the law may be remodeled into a form that does satisfy it. The "principle of stationary action" and the "Hamiltonian differentiation" (5, 139), express such a formulation, from which may be derived the equations of motion, and which remain valid in all reference systems. The characteristic of such a principle is, that it be independent of the coordinates in which it is expressed. For while an equation involving accelerations can obviously not be independent of an acceleration of reference frame, one expressing the minimum of a certain quantity may be thus independent.

To interpret all physical phenomena not as taking place in space, as in an empty container, but as determining the metrical nature of the space; this, we said at the outset, was the underlying motive of relativity. To satisfy it, we may recall that Riemann's method of regarding the element of distance as a constant is not the most general. By letting this element vary as it is carried from point to point and allowing a more general relation,* other than gravitational phenomena may be brought within the metrical scheme. By this method Weyl included electromagnetic phenomena in physical geometry, instead of regarding them as taking place in, but foreign to, the metrical continuum of gravitation. The quantity ϕ_i will give the electromagnetic potentials just as g_{ij} involves the gravitational potential. According to this interpretation, the potentials instead of being arbitrary functions enter the domain of intelligibility and form a smooth bridge between pure mathematics and physics, so that Weyl was led to exclaim: "The dream of Descartes of a purely geometrical physics seems to be progressing to a wonderful fulfillment, although in a manner not foreseen by him." (17, 224.) The concept *mass* has been given a geometrical interpretation as a property of the continuum, although it still retains its empirical nature in that quantitatively it is

* This more general relation would be $ds^2 = l$; $dl = -l\,d\phi$; $\phi = -\log l$; $d\phi = \phi_i\,dx_i$ (17,122).

determinable through the collision of bodies or by weighing, while the vanishing of certain curvature invariants (R_{ij}) determines the nature of the free space surrounding any mass. As an immediate step the non-vanishing of R_{ij} may be postulated as proportional to quantities which express the state of the matter at the point. This step is possible since the convection of energy by material points and the flux of electromagnetic energy have been developed analogously.

Just as Euclidean geometry has validity in the objective world, so has the mechanics of Newton; and as Euclid's geometry is but one in an infinite variety of geometries, so Newtonian mechanics, with only straight line inertial motion, is but a special type of mechanics. Experiment never indicates that a geometry is impossible; it merely aids in determining its relative fitness, or in verifying prediction made in its terms. Prediction is the method of calculating a certain event from a present distribution and then waiting till the event arrives to determine whether the observed event with its probable error covers the calculated event, i. e., whether the calculated event falls within the region circumscribed by the probable error around the mean observed value. Such a prediction is exemplified by the solution of the one-body problem involved in obtaining the equation for the orbit of a planet. In extending the range of predictability, relativity represents a step in the direction of mental adaptation to the conditions of the world. It is a step in the construction of the absolute in which there exists no sequence of cause and effect, but in which all things are spread out in their elemental structure at least so far as their ateleological aspect is concerned. From this way of looking at relativity it is an instrument aiding the mind in adapting itself to the world and in the struggle of ideas for existence represents the survival of the fittest, embracing the rest as limiting cases.

VI

Teleology and the Apriori

The general theory of relativity is so inclusive that it actualizes some of the most ancient speculations; among which, Pythagoras' teaching that the essence of all things is number. Space time has become merely a manifold in which each point event is represented by a set of four numbers; and in which, in spite of transformation of reference system, each event retains its identity, the interval ds between two events remaining invariant in numerical value. As was observed before, what are called the curvature invariants represent nothing that may be pictured. They are numbers characterizing the continuum at each point. They no more express an intuitive image (the like, e. g., of a curved surface), than does the number representing the wave length of a certain color express the sense-quality of that color. Such numbers express something characteristic of the color; all that can be communicated by one experiencing mind to another. By means of these numbers phenomena of light unseen and sound unheard may be predicted, their presence experimentally detected. In this method of science Pythagoreanism had complete sway; numbers give the very essence of the real even in its least perceptible nature. By means of numbers the properties and qualities of the data of experience may be spread out conceptually. Relativity offers a purely conceptual structure, no longer susceptible of being pictured as the sense-world is pictured.

While Kant spoke of space and time as "forms" of intuition or perception, they are rather a numerical frame for perceptions, themselves unperceived. The word perception had better be reserved for a psychological phenomenon allied to sensibility although distinct from it; a certain type of purposive reaction whose stimulus instead of being at or within the surface of an organism (as in sensibility) is outside of it. Space-time has no psychological character, but is a schema presupposed by an experiment indispensable to the task of experiencing. It is a means, common to every consciousness, for individuating, for

identifying the terms of point-event coincidences. Only such coincidences is it possible to record: e. g., the reading on a scale at a certain time, the time-reading itself being a space-reading on another scale. All spatial and temporal quality as distinct has vanished for relativity. Each rests on the same level with arithmetic itself in an elaboration of the category of quantity.

To understand space and time in their relation to perception, this perception as an organism's reactive behavior under the influence of a stimulus existing externally to its boundary must be investigated, and such an investigation is possible only on the postulation of a definition of purposive behavior which distinguishes this type of behavior from the ateleological action of physics.(14)

Space time in its bare quantitative abstractness is the synthesis of the individuating, identifying equipment necessary to single out the events predictable by relativity. This prediction is a union especially of the categories of quantity, causality, and necessity. For physics, it necessarily involves some empirical concept; e. g., *mass*. But *mass* is a term in the expression of a law or classification of a certain reaction; namely, the law of the constancy of ratio of the accelerations imparted to each other by the same two colliding bodies. From a given position of Mercury its position calculated for some future time must necessarily be verified by the event, insofar as the laws of motion are complete, and insofar as all types of phenomena entering into the motion have been taken account of.

Any logical system falls under the category of possibility, and so, of course, must any geometry. Only the application of this geometry to motion of bodies brings it under the category of the actual, and only the prediction of physical events by means of it (cf. the Riemannian geometry and gravitation) brings it within the category of necessity. This latter step, the ancients never made, since they never attained to the conception of inertial motion. Mere consistency in thought means but possibility for thought, not for being, and not at all for necessary being, to the attainment of which a synthesis of rationalism and empiricism is required. Compared with Euclidean geometry, relativity geometry is much more inclusive of experience, since

the empirical concept mass and energy are given a geometrical interpretation.

Characteristic of the advance in the predictability of nature is the extension and adaptation of logic to the data of experiment. Phenomena thus become certain space-time configurations and the relation of cause and effect a functional and hence a logical relation. The real being, that which satisfies the functional equations, hereby obtains its rational nature. The concept which distinctly involves the category of cause and effect is that of force, or mass times acceleration; but through relativity even such a concept has become kinematical, so that the approach to the real is not a copying of data but a slight bending, stretchingand distorting of the schema resulting from the application of the categories to space-time, with the result that the real falls in line with thought; or thought progressively informs the real.

The variation of the element of distance $dl = l\phi_i dx_i$ (where ϕ_i introduces the electromagnetic potential) is a means of further modifying the metrical field so that the electromagnetic lines of force along with those of gravity combine into extremals of the field, wherewith the paths of all freely moving bodies become extremals of the space manifold. This method represents the adaptability of the conceptual forms, which relieving space of its old obligation to be homogeneous, may explain and make intelligible fields and charges as modifications of metrics. If Hegel meant anything by interpreting Kant's categories in a less rigid and more dynamic style, relativity in extending the notion of manifold to gravitational and electromagnetic fields may be said to have accomplished a Kantian task in a post-Kantian spirit.

While relativity as the name implies has laid stress on the symmetry of relations, an equilibrium between systems, it has at the same time given a new insight into "the absolute," into the ultimate goal of ateleological science. Relativity has all the characteristics of a study which possesses objective universality and typifies a Hegelian dialectical movement in reconciling contradictions and this reconciliation always leads to an advance in power of prediction.

With the idea of functional relation, the category of causation

intimately connected with the unique direction of time-progress from past to future has been brought into clearer light. The category of causality requires that, under the hypothesis of a law, the cause is the necessary and sufficient condition for the effect. A force applied to an unobstructed object is necessary and sufficient to alter its velocity. The force of gravity turns bodies out of their straight line paths. As such, however, the force is, if not unintelligible, at least an undesirable complication. The observations of what bodies do never forces its elimination*; but in an apriori construction (like that of a curved manifold) the "mystical" force acting at a distance becomes at once intelligible by being made meaningless. The problem of the nature of the force acting across empty space has been solved by being shown to be no problem. A causal connection has been replaced by a functional relation in which the space-time coordinates of a body are functions of its world-line; that is, by the time indicated by a clock on the body. The step was made possible not by analytic refinement of a theory already in existence but by the carrying through of a new principle obtained by a reflective standpoint. More inclusive in its view, this standpoint embraced the most fundamental "physical" notions, inertia, gravitation, potentials, together with the geometrical conception of a mobile, and yielding metrical field. The first suggestion of analogy ended in a demonstration of identity.

Relativity is a synthesis of the work of Ancients, Copernicus, Kepler, Descartes, Galileo, Newton, Laplace, Gauss, Riemann and Mach. But the work of each of these expresses a reflection upon the work of the predecessors rather than a simple analytic continuation. The resultant synthesis is not, however, a blurring but a unification in which preceding definitions and propositions are retained in their essential characteristics. The equalizing of all reference frames means an emphasizing of the positive aspect of a negative, as when a non-

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* The definition of a potential function as $V = \frac{m}{r} + \frac{mh^2}{r^3}$ gives the same orbit as relativity, compare (5, 86) and as an *actio distans* law reveals how choice enters in the interpretation of "facts." The formal advantages of the theory of relativity are, however, unquestionable.

stationary or moving system may equally well be regarded as stationary. All centers of view are equally valid and represent a complete dialectic. At the same time the law of transformation results in a unification that gives the dialectic a positive character instead of a merely negative one ending in skepticism. The clearing away of old prejudices connected with an attachment to Euclid and Newton as the ultimate standard is a legitimate task for doubt and skepticism, but the construction of a grander structure in which these are elements typifies the critical constructive attitude of thought, which does not stop with showing that sense experience is a flux, but reveals whole systems of thought to be but stages in the phenomonology of culture.

The dialectic of principles according to which what is straight may legitimately from the point of view of another system appear curved is a dialectic of understanding analogous to the development of a different geometry by negating a certain axiom and deducing the consequence. This dialectic is distinct from the dialectic of reason which proceeds according to a different kind of law, namely that of *purpose* or *end*. (14) Here need be no mechanical likeness between the causes producing the same purpose-effect and where the other results following upon the means have no property in common except that they all exist in a mechanical medium. A *means* is always necessary for the occurrence as *end* but never sufficient, for if it were also sufficient the effect would follow necessarily upon the means as cause and the situation would not differ from that of ordinary mechanical cause and effect relation, in which the effect follows invariably on the cause. Certain effects of the means which being exceptions do not fulfill the purpose may on the basis of another classification fulfill the definition of purpose. Here there is involved the dialectic of ends, the true dialectic of life and reason, in contrast with which the dialectic of relativity is one of understanding. Any dialectic, however, strictly speaking, involves choice and hence belongs to the categories of life, mind, and consciousness. The dialectic movement is one in which a purpose fulfilled becomes a new means for another purpose. The successive reflective standpoints mentioned in

the development of relativity are just such a movement of mental adaptation to the world: a labor of setting up one-to-one a correspondence between thought and reality, of a carrying out of the purposes of life amid increasingly complex mechanical situations. The possibility of predicting what mechanical systems will do, even those which reveal no purpose, expresses a teleological aspect of life in its conquest of nature. Each new generalization means increased knowledge and power in controlling the phenomenon, and control presupposes a knowing of what will happen.

The symbols with which the physicist works involve the logic of understanding, but the goals and ends which he sets before him and to the realization of which the symbols and the logic are but means, are objects of a "rational will." The symbols fix and communicate his thoughts; but as he works with these he is continually required to make choices and exercise a freedom of interpretation. What he starts out with, his presuppositions, are at least partly subject to choice. The axioms of principles which he posits as apriori are so by choice. In the special theory the two postulates were set up by Einstein after a procedure of selection from a mass of empirical material through an act of free choice. The choice gave the way for carrying through the same principle of relativity in two domains. In that way he succeeded where others failed.

After a definition of purpose has been set up, a definition which includes all types of purposive activity both in "works of art" and in "products of nature," life may be defined as that type of behavior whose purpose is *self-maintenance;* by which we understand a behavior whose gestures are recognized as taking place for the sake of taking place *again.* From its very definition, this kind of behavior is cyclical: we eat, drink, and sleep that we may eat, drink, and sleep *again.* (14) Such is life reduced to a minimum. If each of these acts takes place in order that the others may take place the life is organized. These characteristics of life are shared in common by an amoeba and an Einstein. The difference in level denoted by the phrases *lower life* and *higher life* may be defined in terms of variety of purpose and their attainment. The greater the chance of attained

purpose the higher is the life. Any judgment of relative value must rest on such conceptions.

In the working out of these definitions in reality, it must be remembered that choice enters especially in developing a new theory; for more than one interpretation being empirically possible, the one to be selected is the one best fitted to attain the ends in view. Relativity as a successful theory was made possible as the result of free choice in the setting up and carrying through of a definite principle. A psychology of invention and discovery must primarily consider ends and means. These are the frame work of life within which discursive logic has but the rôle of analytically defining concepts.

A regulative principle, to strive for maximum unity, sets up before the mind the ideals of unifying or carrying out analogously the description of various fields; e. g., electrodynamics and gravitation. Before a theory has succeeded, such notions are mere ideals within the worker's consciousness; but their enormous importance becomes apparent when a few individuals have given them empirical application. A slight discrepancy of 43″ per century in the advance of the perihelion of Mercury could not be explained by an *ad hoc* assumption of unseen or imaginary masses. In the future there will appear more and more of these slight shifts in the predicted distribution of point events. In all such cases, the only way to bring apparent exception into uniformity is to reformulate the first principles. The empirical considerations in favor of relativity involve very slight quantities; but they were sufficient to invite a decided revolution in elementary concepts; whereby we see that the geometry of manifolds is not merely a logical possibility, but has properties which may be employed to define empirical concepts (mass, energy). The manifold in being thus brought in touch with experiment *has been made real:* at the same time, reality has been made more unified. The interpretation of mass as a geometrical property raises the concept into this class of geometrical relationships, wherewith it partakes of the universality of geometry while the apriori has been developed and extended. The communicability between the various disciplines of the "exact"

sciences; the making intelligible of the potentials as determining the metrical groundform; these are the formal results obtained by the apriori reformulation.

The conc'usion seems unavoidable that relativity offers a method enabling science after experiencing objects in one space-time setting to change the setting and begin experimenting anew. If Newton's law of gravitation represents only the first term in an endless series, the "law" of which may not be known before the "probable error" of all measurements has been reduced to zero, every shift in the empirical distribution of events will invite a generalization of the metrical continuum. But to *impose* a new individuating form it must first be *made*, and the reflection that this apriori equipment may always be *remade* to aid further experiment confirms the *Critical Theory* in its modern extension.

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